ON THE MOTION OF A VISCOUS ELECTRICALLY-CONDUCTING LIQUID UNDER THE ACTION OF A ROTATING DISK IN THE PRESENCE OF A MAGNETIC FIELD

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PMM Vol.24, No.5, 1960, pp. 906-908

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(Received 14 July 1960)

In 1921 Karman [1] considered the problem of the motion of a viscous incompressible fluid under the influence of an infinite plane disk, uniformly rotating around its axis. For this case the Navier-Stokes equations can be reduced to a system of ordinary differential equations with a single independent variable. Given below is a generalization of the Karman problem to the case of the motion of a liquid having a finite electrical conductivity, in the presence of a magnetic field which is uniform and perpendicular to the plane of the disk at infinity. The electrical conductivity of the disk is assumed equal to zero.

1. The magneto-hydrodynamic equations for a viscous incompressible fluid with finite electrical conductivity σ in the steady state have the form [2]

$$\operatorname{div} \mathbf{H} = 0, \qquad \operatorname{div} \mathbf{V} = 0, \qquad (\mathbf{V}\nabla)\mathbf{H} = \mathbf{H}(\nabla\mathbf{V}) + \frac{1}{4\pi\sigma}\Delta\mathbf{H} \tag{1.1}$$

$$(\mathbf{V}\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla\left(p + \frac{H^2}{8\pi}\right) + \frac{1}{4\pi\rho}(\mathbf{H}\nabla)\mathbf{H} + \nu\Delta\mathbf{V}$$

Here **V** is the velocity vector of the liquid, **H** is the magnetic field intensity, p is the pressure, ρ is the density and ν is the kinematic viscosity.

We introduce a system of cylindrical coordinates r, ϕ and z, in which the plane $z = 0$ coincides with the plane of the disk. The angular velocity of the rotating disk is designated by Ω. The boundary conditions of the problem have the form

$$V_r = 0, \quad V_\varphi = \Omega r, \quad V_z = 0 \qquad \text{at } z = 0$$
$$V_r = 0, \quad V_\varphi = 0, \quad H_r = 0, \quad H_\varphi = 0, \quad H_z = H_0 \qquad \text{at } z = \pm\infty \tag{1.2}$$

The axial component of the velocity does not vanish at infinity, but tends toward some limiting value subject to determination. This is because of the fact that, inasmuch as the liquid near the disk moves radially under the action of a centrifugal force, in order to insure continuity there must exist a steady influx from infinity in the direction of the disk. Since the magnetic field is constant and a change in sign of the field at infinity does not change the form of Equations (1.1) it follows that, taking account of the fact that H_z is an even function and H_r and H_ϕ are odd with respect to z, it is possible to write the boundary conditions for the components of the field in the plane of the disk

$$H_r = 0, \quad H_\varphi = 0 \qquad \text{at } z = 0 \tag{1.3}$$

The magnitude of the axial component of the field H_z at $z = 0$ is subject to determination. Thus the problem reduces to a consideration of the flow in one (the upper, say) half-plane. We shall seek a solution in the form

$$V_r = \Omega r u(\zeta), \quad V_\varphi = \Omega r v(\zeta), \quad V_z = \sqrt{\Omega\nu}\, w(\zeta)$$
$$H_r = \sqrt{4\pi\rho}\,\Omega r f(\zeta), \quad H_\varphi = \sqrt{4\pi\rho}\,\Omega r g(\zeta), \quad H_z = \sqrt{4\pi\rho\Omega\nu}\, h(\zeta) \tag{1.4}$$
$$p + \frac{H_r^2 + H_\varphi^2 + H_z^2}{8\pi} = -\rho\Omega\nu P(\zeta) \qquad \left(\zeta = \sqrt{\frac{\Omega}{\nu}}\, z\right)$$

The radial and tangential components of the velocity and of the magnetic field are assumed here to be proportional to the distance from the axis of the rotation disk, and the vertical components V_z and H_z and the total pressure (the sum of the hydrodynamic and magnetic pressures) are constant throughout each horizontal plane. Projecting (1.1) on the axis of the cylindrical coordinates and substituting (1.4), we obtain a system of ordinary differential equations for dimensionless variables

$$\begin{aligned} &h' + 2f = 0, && w' + 2u = 0 \qquad (k = 1/4\pi\sigma\nu) \\ &kf'' + hu' = wf', && u'' + hf' + f^2 - g^2 = wu' + u^2 - v^2 \\ &kg'' + hv' = wg', && v'' + hg' + 2fg = wv' + 2uv \\ &kh'' + hw' = wh', && P' + hh' + w'' = ww' \end{aligned} \tag{1.5}$$

Here, the prime indicates differentiation with respect to ζ. The last of these equations can be integrated:

$$P + w' + \frac{h^2 - w^2}{2} = \text{const} \tag{1.6}$$

It is clear that the total pressure P is found to be a constant independent determined from the remaining equations. We notice here that the fourth of these equations (1.5) can be written in the form

$$kf' + hu = wf \tag{1.7}$$

by use of the first two.

The third equation of (1.5) can be obtained from (1.7) and the first two equations of (1.5). Therefore, the final system of equations which are to be integrated can be written in the form

$$\begin{aligned} &h' + 2f = 0, \qquad kf' + hu = wf \\ &w' + 2u = 0, \qquad kg'' + hv' = wg' \\ &u'' + hf' + f^2 - g^2 = wu' + u^2 - v^2 \\ &v'' + hg' + 2fg = wv' + 2uv \end{aligned} \tag{1.8}$$

The boundary equations, on the basis of (1.2), (1.3) and (1.4), take the form

$$\begin{aligned} &u = 0, \quad v = 1, \quad w = 0, \quad f = 0, \quad g = 0 \qquad \text{for } \zeta = 0 \\ &u = 0, \quad f = 0, \quad h = \chi = \frac{H_0}{\sqrt{4\pi\rho\Omega\nu}} \qquad \text{for } \zeta = \infty \\ &v = 0, \quad g = 0, \end{aligned} \tag{1.9}$$

The number of these conditions exceeds the order of the system (1.8) because the point at infinity is a singular point.

2. We investigate the asymptotic behaviour of the system (1.9) as ζ approaches ∞. We designate

$$w(\infty) = -\lambda \tag{2.1}$$

Then the solution in the neighborhood of the point at infinity can be put in the form of an expansion in exponentials

$$e^{-\alpha_1\lambda\zeta}, \; e^{-\alpha_2\lambda\zeta} \qquad \left(\alpha_{1,2} = \frac{1+k}{2k} \pm \sqrt{\left(\frac{1-k}{2k}\right)^2 + \frac{1}{k}\,\frac{\chi^2}{\lambda^2}}\right) \tag{2.2}$$

The main terms of the expansion have the form

$$\begin{aligned} &f = A_1 e^{-\alpha_1\lambda\zeta} + A_2 e^{-\alpha_2\lambda\zeta} + \ldots, \qquad g = B_1 e^{-\alpha_1\lambda\zeta} + B_2 e^{-\alpha_2\lambda\zeta} + \ldots \\ &h = \chi + \frac{2A_1}{\alpha_1\lambda} e^{-\alpha_1\lambda\zeta} + \frac{2A_2}{\alpha_2\lambda} e^{-\alpha_2\lambda\zeta} + \ldots \\ &u = A_1 \frac{\lambda}{\chi} (k\alpha_1 - 1) e^{-\alpha_1\lambda\zeta} + A_2 \frac{\lambda}{\chi} (k\alpha_2 - 1) e^{-\alpha_2\lambda\zeta} + \ldots \\ &v = B_1 \frac{\lambda}{\chi} (k\alpha_1 - 1) e^{-\alpha_1\lambda\zeta} + B_2 \frac{\lambda}{\chi} (k\alpha_2 - 1) e^{-\alpha_2\lambda\zeta} + \ldots \\ &w = -\lambda + \frac{2A_1}{\alpha_1\chi} (k\alpha_1 - 1) e^{-\alpha_1\lambda\zeta} + \frac{2A_2}{\alpha_2\chi} (k\alpha_2 - 1) e^{-\alpha_2\lambda\zeta} + \ldots \end{aligned} \tag{2.3}$$

Here A_1, A_2, B_1 and B_2 are arbitrary constants subject to determination. In addition to λ we have five such constants in conformity with the number of boundary conditions for $\zeta = 0$. From the expression (2.2) for the coefficients α_1, α_2 it follows that the disturbance of the magnetic field, and the velocity field die out, and the solution of the form considered exists only when

$$S = \frac{\chi^2}{\lambda^2} = \frac{H_0^2}{4\pi\rho V_{z\infty}^2} < 1 \tag{2.4}$$

The magnitude S determines the ratio of the magnetic force, characterized by the stress $H^2{}_0/4\pi$, to the inertial force $\rho V_{z\infty}{}^2$, or the ratio of the density of magnetic energy $H_0{}^2/8\pi$ to the density of kinetic energy of the material, $1/2\,\rho\, V_{z\infty}{}^2$. At the same time it is equal to the square of the ratio of the velocity of propagation of the Alfven waves to the velocity of the current. If the intensity of the magnetic field at infinity is sufficiently large so that $S > 1$, a disturbance of a wave type may be propagated an indefinite distance from the disk; so that flow of this type may be considered physically impossible. Thus the region of existence of the desired solution is limited to the interval of values of the parameter $0 < S < 1$.

We consider now the behavior of the solution near $\zeta \to 0$. The principal terms of the expansion of the desired function in the neighborhood of the point $\zeta = 0$ have the form

$$u = u'(0)\,\zeta + \ldots, \qquad v = v'(0)\,\zeta + \ldots, \qquad w = -u'(0)\,\zeta^2 + \ldots \tag{2.5}$$

$$f = -\frac{1}{2k}\,h(0)\,u'(0)\,\zeta^2 + \ldots, \qquad g = g'(0)\,\zeta + \ldots, \qquad h = h(0) + \frac{1}{3k}\,h(0)\,u'(0)\,\zeta^3 + \ldots$$

Since $u'(0) > 0$, $h(0) > 0$, it follows that at the surface of the disk the radial component of the magnetic field is negative and the axial component has a minimum. Thus it can be assumed that the magnetic lines of the axisymmetrical part of the field $\mathbf{H}_a = (H_r, H_z)$ diverge as they approach the disk.

3. The solution of the system of equations (1.8) may be obtained by a numerical method. However, this presents considerable difficulty in view of the large number of boundary conditions (1.9). Therefore, there is some interest in the consideration of approximate methods of solution. One of these methods may be based on the assumption that the parameter

$$k = \frac{1}{4\pi\sigma\nu} = \frac{\eta}{\nu} \tag{3.1}$$

which characterizes the ratio of the electrical resistivity to the viscosity of the liquid, is large compared to unity. This assumption is usually made; thus, for mercury, the magnitude of k is of the order 6×10^6. For $k \gg 1$, the flow region may be divided into two parts: the viscous boundary layer at the surface of the disk, with thickness $\delta_1 \sim 1$, and the external stream in which the influence of the finite electrical conductivity is greater than that of the viscosity; this has a characteristic normal dimension $\delta_2 \sim k$. The calculation of the flow in these regions can be carried out by a method of successive approximations in which the original condition for computing the boundary layer is the velocity field in the absence of a magnetic field, already known from solutions obtained in [9,3]. The system of equations of the first approximation for the outer portion of the current differs from (1.8) by the absence of the second derivative of the velocity in the last two equations, since their ratio to the remaining terms will be of order $1/k$. This corresponds to the fact that the role of viscosity is unimportant here. The solutions obtained must be matched along the outer boundary of the boundary layer; in other words, the boundary conditions for the computation of the outer part of the flow are determined by the asymptotic values of the variables in the boundary layer. In practice, the boundary-value problems for both regions may be approximately solved, for example by integral methods [1].

4. In conclusion, we shall derive the expressions for the components of the vectors of current density and electric field intensity, which can be determined on the basis of the equations [2]

$$\mathbf{j} = \frac{1}{4\pi} \operatorname{rot} \mathbf{H}, \qquad E = \frac{1}{\sigma} \mathbf{j} + [\mathbf{H}, \mathbf{V}] \tag{4.1}$$

Using the solution of (1.4), we obtain

$$j_r = -\sqrt{\frac{\rho}{4\pi\nu}}\,\Omega^{3/2} r g', \qquad j_\psi = \sqrt{\frac{\rho}{4\pi\nu}}\,\Omega^{3/2} r f', \qquad j_z = \sqrt{\frac{\rho}{\pi}}\,\Omega g \tag{4.2}$$

$$E_r = \sqrt{4\pi\rho\nu}\,\Omega^{3/2}(r - kg' + wg - vh), \qquad E_\psi = 0$$

$$E_z = \sqrt{4\pi\rho}\,\Omega^2 r^2 (vf - ug) + 2k\sqrt{4\pi\rho}\,\Omega\nu g \tag{4.3}$$

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Translated by D.T.W.